Shutout Games on Graphs

ALEXANDRU CIOBA MICHAIL SAVVAS

Abstract

Two players take it in turn to claim edges from a graph G. The first player ("Maker") wins if at any point he has claimed s edges at a vertex without the second player ("Breaker") having claimed a single edge at that vertex. If, by the end of play, this does not occur we say that Breaker wins. Our main aim is to show that for every s there is a graph G in which Maker has a winning strategy.

1. Introduction

In this paper we prove a result concerning a game on graphs that is a natural problem in the context of combinatorial game theory. The statement of the theorem is as follows:

Theorem 1.1 *Suppose that two players (Player 1 - "Maker" and Player 2 - "Breaker") take turns to colour edges of a graph. Then, for any integer $s > 1$ there is a graph G in which Maker can colour s edges at a vertex before Breaker has coloured any at that vertex.*

The background behind this result is that of Maker-Breaker games. We proceed to give a short, self-contained account of the definitions involved. For additional theory and motivation see [1].

Definition 1.2 (Maker-Breaker game) *Let X be any finite set. By a Maker-Breaker game on X we mean a collection of subsets of X, $H \subseteq \mathcal{P}(X)$, together with the following rule of play: two players, Maker and Breaker, take turns colouring points of X. Maker wins if after all the points of X have been coloured, one of the elements of H has been coloured entirely by him. Otherwise Breaker wins. X is called the **board** and H the set of **winning lines**.*

To any game on a board X we may associate a different game, via the following definition:

Definition 1.3 (Shutout game) *Maker wins the shutout game with parameter b on the board X with set of winning lines H if he can colour b points of a winning line in H without Breaker having played in that line.*

For an arbitrary graph G, we consider the Maker-Breaker game, where Maker tries to colour all the edges incident to a vertex of G: The board is $X = E(G)$ and the set of winning lines H consists of the edge neighbourhoods of the vertices of G.
We wish to study the associated shutout problem. Thus we are interested in the following:

Definition 1.4 (Shutout game with parameter s) *We say that Maker wins the shutout game with parameter s on a graph G if he colours s edges at a vertex without Breaker having played at that vertex.*

Finding a graph where Maker can win the shutout game with parameter s is very much

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dependent on the density and the distribution of the edges in the graph. A large number of edges means that Maker can achieve more complex structures faster, but it also gives room to Breaker to touch as many vertices as possible in a short amount of time.

Observe that on the complete graph Maker cannot win for non-trivial values of s. It is not difficult to see that Breaker can play in a way such that after an even number of turns, among all vertices touched by Maker, there is at most one which is untouched by Breaker, and furthermore, the one vertex has at most one adjacent edge coloured by Maker. This is done by an easy induction on the number n of rounds played (a round consists of a turn of both players). During the first round, if Maker colours the edge xy, Breaker colours edge xz for some z. Suppose the claim is true for n with the only candidate vertex not touched by Breaker being u together with the edge uv coloured by Maker. Let S denote the set of vertices touched by Breaker (note that $v \in S$). Let ab be the edge coloured by Maker at turn $2n+1$. If a, b do not belong to $S \cup \{u\}$, then Breaker colours ua. If one belongs to $S \cup \{u\}$ but the other does not, then Breaker plays edge uc for some c. In both cases it is clear that we arrive at the desired state.
Thus to produce a graph G as stated in Theorem 1.1 one has to exercise a certain measure of control over both the number of edges in relation to the number of vertices but also over the geometry of the graph.

In the next section, we proceed to the proof of Theorem 1.1 and investigate some further properties of these games. Finally, we conclude by drawing a connection with another class of Maker-Breaker games and formulating some conjectures about the behaviour of the games described.

2. Main Results

With the definitions above, our main theorem can be restated as follows:

Theorem 1.1 *For any $s \geq 1$ there exists a graph G such that Maker wins the shutout game on G with parameter s.*

To facilitate our discussion, we introduce a piece of terminology: we call a monochromatic $K_{1,s}$ at a vertex v a **free s-star** if after Breaker's turn no edge incident to v has been played by Breaker. We say that if at any point Maker achieves a free s-star, then he has won the free s-star game on the graph. Clearly winning a free s-star game is the same as winning the shutout game with parameter $s+1$.

We prove Theorem 1.1 by induction. However, Maker being able to win the shutout game at a single vertex does not suffice for the inductive step and something more is required. The key idea that makes the induction work is to strengthen the claim by requiring that Maker wins at multiple vertices. This result takes the following formulation:

Theorem 2.1 *For any $s \geq 1$, $n \geq 1$ there exists a graph G with $\delta(G) > s$ such that Maker can achieve at least n free s-stars at n independent vertices when playing on G.*

Clearly Theorem 2.1 implies Theorem 1.1, as the latter is a special case of the former.

We will say that at some stage of play an edge is **untouched** if it has not been coloured by Maker or Breaker and similarly for vertices (no edge incident to it has been coloured) or other structures (e.g. graphs) in the obvious way. For example, a subgraph will be

untouched if no edges with at least one endpoint in it are coloured.
For technical reasons, we prove a stronger claim by induction on s:

Theorem 2.1* *Theorem 2.1 still holds if we demand that Maker can win without saturating a vertex, i.e. without needing to colour all the edges incident to any vertex.*

Proof of Theorem 2.1* We induct on s.

Clearly for the base case $s = 1$ either a large cycle graph or a large collection of disjoint triangles suffices. Indeed, in the latter case, Maker colours an edge of an untouched triangle on every turn, thus creating two new free 1-stars. Breaker has the option of either playing one of the two adjacent edges in some triangle which Maker has touched, in which case Maker will have gained a free 1-star at the end of Breaker's turn, or, otherwise, Breaker can play in some untouched triangle, in which case Maker will have gained two free 1-stars at the end of Breaker's turn. So at the end of each round of play Maker has gained at least one free 1-star. Taking $4n$ independent triangles to start with, we see that Maker can always play in some untouched triangle in his first $2n$ turns and so at the end of the $2n$-th round, he will have made at least $2n$ free 1-stars, with at least half at independent vertices (and Maker has not used up all the edges at any vertex), as desired.

For the inductive step, assume n is given and suppose that G_s^k denotes a graph where Maker can make k free s-stars at independent vertices without saturating any vertex (see Figure 1). Let I_N^m be disjoint sets of $N(\geq 1)$ independent vertices indexed by $m = 1, \dots, \binom{|G_s^k|}{2s+2}$. We join each subset of size $2s+2$ of G_s^k to an I_N^m completely, i.e. adding an edge joining any two vertices that do not belong to the same graph out of the two; we call these **cross-edges** and the resulting graph $\tilde{G}$. Note that the degree of every vertex in some I_N^m is $2s+2$, while the degree of the vertices in G_s^k is at least $\delta(G)+N > s+1$ (recall that by the inductive hypothesis $\delta(G) > s$), so that $\delta(\tilde{G}) > s+1$.

Claim *For sufficiently large k and N (see end of proof) Maker can make n free $(s+1)$-stars at n independent vertices of $\tilde{G}$.*

This clearly establishes the inductive step and will conclude the proof.

We now present Maker's strategy for the shutout game on $\tilde{G}$.
Firstly, Maker plays in the G_s^k subcopy of $\tilde{G}$ according to the winning strategy that grants him k free s-stars at independent vertices. In this way Maker can still win k free s-stars in $\tilde{G}$, since if there were a chance for Breaker to prevent this in $\tilde{G}$ he could have also done it in G_s^k by playing an edge at the same vertex he would play on $\tilde{G}$ (note that there is no point for Breaker to play a cross-edge twice at a vertex and the fact that Maker does not need to saturate any vertex allows for this).
Suppose hence that after T_0 moves by both players Maker has achieved k free s-stars at a set U of independent vertices of the G_s^k subcopy inside $\tilde{G}$. Let us choose $k = l(2s+2)$ and partition U into l subsets of cardinality $(2s+2)$ $U_1, \dots, U_l$ in any way, for l sufficiently large (see end). Let $I_N^1, \dots, I_N^l$ denote the sets of independent vertices appended to the U_i as described above.

Now Maker plays according to the following algorithm (regardless of what Breaker does in particular):

Step 1 Find a U_i none of whose vertices has been touched by Breaker.
Step 2 Find $v \in I_N^i$ untouched by Breaker.
Step 3 Find a vertex $u \in U_i$ which is untouched by Breaker and play the cross-edge vu.
Step 4 Repeat Step 3 $(s+1)$ times.
Step 5 Go to Step 1 and repeat $2n$ times.

Assuming we can iterate this procedure $2n$ times for l, N large enough (again see end), we will show that after $2n$ iterations Maker will have made at least $2n$ free $(s+1)$-stars.

- Firstly, from the above description it is clear that in each round of play Maker creates at least one new $(s+1)$-star at some vertex of U.
- Moreover, by the independence properties of the vertices that arise (U_i, I_N^i are sets of independent vertices and also completely independent from other U_j, I_N^j by construction), Breaker can never destroy more than one free $(s+1)$-star in his turn, since no two such vertices are joined by an edge.
- We now come to a crucial observation: Notice that Maker's strategy consists of playing $(s+1)$ times "inside" a pair (U_i, I_N^i) for $2n$ choices of such pairs. If Breaker elects to always destroy the new $(s+1)$-star created by Maker in U_i immediately after it is formed by playing some edge at that vertex, then at the $(s+1)$-th repetition of Step 3 Maker will create two free $(s+1)$-stars at once, one inside U_i and the other at v, as v will have remained untouched by Breaker this whole time and will be joined with $(s+1)$ vertices of U_i. Therefore, during these $(s+1)$ rounds of play, there is a round when either Maker creates two free $(s+1)$-stars or Breaker does not destroy a free $(s+1)$-star.

These three observations together show that after $(s+1)$ rounds the number of free $(s+1)$-stars that Maker has achieved increases by at least 1. Clearly thus after $2n$ iterations of this procedure Maker will have achieved at least $2n$ free $(s+1)$-stars in $\tilde{G}$. These lie inside U and the I_N^i's, so at least half of them will be inside one of the two and hence independent. This shows that Maker will have achieved n free $(s+1)$-stars at independent vertices, as wanted.

We remark now that Maker does not saturate any vertex in this process. Clearly any vertex of G_s^k will not be saturated, as Maker uses at most one cross-edge for each of these vertices and their degree in $\tilde{G}$ increases by at least N. For the vertices in the I_N^m's Maker uses at most $s+1$ edges, whereas their degree equals $s+2$.

Hence, in order to complete the proof, we have to pick concrete values for l, N that allow everything in the above to go through.
We notice that the total number of turns required by the above description (assuming there is enough space to follow the outlined strategy) equals $T = T_0 + 4n(s+1)$, since T_0 turns are needed so that Maker wins in the subcopy of G_s^k inside $\tilde{G}$ and then $2n(s+1)$ rounds so that Maker achieves the $2n$ free $(s+1)$-stars. Therefore, if we take $l > T + 2n$ Maker will be able to find $2n$ untouched U_i by Breaker (note that independence plays a role here too). Similarly for $N > T + 1$ there will always be some vertex of I_N^i which has not been touched by Breaker. We also note that for the iteration of Step 3, Maker can always find consecutively $(s+1)$ vertices of U_i untouched by Breaker, since they both start with U_i untouched and $|U_i| = 2s+2$, so Breaker can touch at most $(s+1)$ of these in $(s+1)$ rounds of play.
These estimates conclude the proof. $\square$

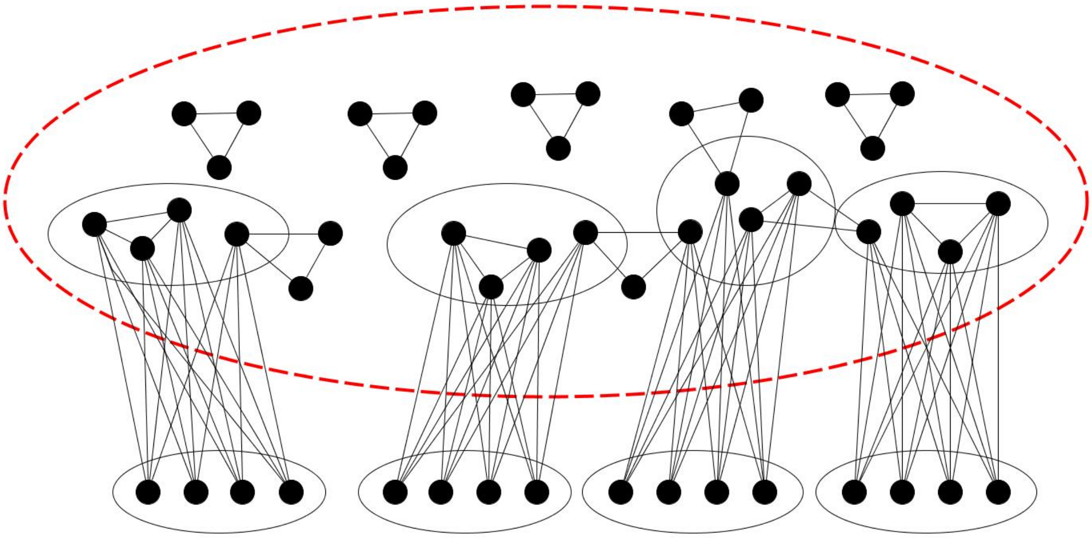

Figure 1: An illustration of part of G_2^{11}.

Theorem 2.1 prompts the following question: how small can the maximal degree of a graph be for Maker to be able to win the shutout game? Furthermore, how small can the degree of a vertex in any graph be, whilst allowing Maker to win a free s-star at this vertex? From now on, when we say that Maker may win a free s-star at a vertex v, we mean that v belongs to a minimal family of vertices, such that there exists a strategy for Maker which guarantees that he wins the free s-star game at an element of the family. By a **minimal family** we mean a set S of vertices with the following property: if a game is played requiring Maker to complete a free s-star at any vertex in a specified set, then for the set $S \setminus \{v\}$, for any $v \in S$, this game is a Breaker win.

Definition 2.2 *The Local Shutout Number of a positive integer s, denoted by* $\mathrm{LS}(s)$*, represents the least maximal degree of a minimal family S, across all graphs, at which Maker may win a free s-star. Similarly the Global Shutout Number,* $\mathrm{GS}(s)$ *represents the smallest maximal degree of a graph G where Maker may win a free s-star.*

Knowledge of the rate of increase of these numbers may prove essential in determining strategies for Breaker on certain graphs. As one might hope, the two numbers defined above coincide.

Lemma 2.3 *The Shutout Numbers satisfy* $\mathrm{LS}(s) = \mathrm{GS}(s)$.

Proof We start our proof with the following observation. Suppose Maker wins a free s-star at a vertex v, in a graph G. Then, the removal of an edge from v cannot prevent Maker from winning a free $(s-1)$-star at v (Maker follows the same winning strategy, except that he passes if he needs to play the removed edge). Also, note that $\mathrm{LS}(s) \leq \mathrm{GS}(s)$. Suppose G is edge minimal such that we can win the free s-star game on G and we can do so by realizing the Local Shutout Number at a vertex. Let $\Delta(G) = m > n = \mathrm{LS}(s)$. Then every vertex v_0 of degree m is surrounded by vertices of degree n only since otherwise we could remove an edge from v_0 and still win the free s-star game (in the same way as before, just passing instead of playing that edge if need be). Hence, at each such vertex, we can disconnect an edge as shown in the figure (Figure 2) and still be guaranteed to win the free s-star game since we are a priori granted that we can do so at a vertex of degree n. Thus we can successively lower the maximal degree to n whence $\mathrm{LS}(s) = \mathrm{GS}(s)$. $\square$

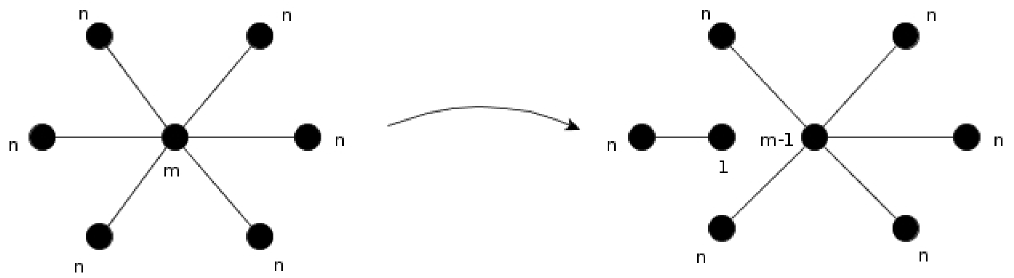

Figure 2: Disconnecting an edge.

Since these two quantities coincide we merge the two definitions and what we will simply call the Shutout Number from now on we denote by $\sigma(s)$.
Now observe that in the construction in Theorem 2.1* $\Delta - \delta$ is potentially quite large. This is however not typical or necessary in general. The previous proof constructs a graph in which $\sigma(s) = \Delta(G)$. This, coupled with the simple observation that given a graph G such that we have $\Delta = \Delta(G)$, there is a Δ-regular graph H such that $G \leq H$ and the copy of G inside H can be obtained from H by just removing vertices, allows us to deduce that if we limit ourselves to playing the shutout game on regular graphs, we do not restrain the generality of the problem. Indeed, regular graphs are extremal examples, when it comes to winning the shutout game.

Finally, we have the following corollary.

Corollary 2.4 *The Shutout Number $\sigma(s)$ is strictly increasing.*

Proof Suppose G is a m-regular graph, where $m = \sigma(s)$. Run the following algorithm on G: Pick a vertex v_0 of degree m. Disconnect an edge from v_0 as previously, changing its degree to $m-1$, and leaving the other vertex neighbourhoods unchanged. Repeat until there are no more vertices of degree m. Then, by an earlier observation, we can win the free $(s-1)$-star game on this new graph, and its maximum degree is $m-1$. So $\sigma(s-1) \leq \sigma(s) - 1$. $\square$

3. Conclusions and Open Questions

In order to study quantitative properties of the Shutout Numbers it is useful to introduce another class of Maker-Breaker games, which are pertinent to the shutout game we have been examining: For a regular graph G of degree m and a natural number s consider the Maker-Breaker game with board $X = E(G)$ and set of winning lines H consisting of the subgraphs of G isomorphic to $K_{1,s}$ (s-stars). We refer to this game as the game on G with parameters (m, s) or an (m, s)-game on G as a shorthand.

Given a regular graph G of degree m where Maker can achieve a free s-star at a vertex v, it is clear that he may win the game on G with parameters $(m, s + \lceil \frac{m-s}{2} \rceil)$ by just playing at this vertex after winning the free s-star. This strategy might seem crude, but in most cases a better approach for Maker is not known. Crucial to this strategy is the rate of increase in the shutout numbers, $\sigma(s)$, which act as a type of complexity parameter of the game. For example, in a $(7, 6)$-game it would work if we knew that $\sigma(4) \leq 7$. This however remains unknown. It is clear though that the strategy proves useful for the $(7, 5)$-game where only a free 2-star is required by Maker for this strategy to apply and it can be shown that $\sigma(2) = 3$. Furthermore, one can give a rudimentary lower bound for the Shutout Numbers as follows:

Lemma 3.1 *The Shutout Number satisfies* $\sigma(s) \geq 2s - 3$.

Proof If Maker achieves a free s-star at a vertex of an m-regular graph, by the strategy outlined above he can win the game on G with parameters $(m, s + \lceil \frac{m-s}{2} \rceil)$. This, in turn, implies that $m > \frac{4}{3}(s + \lceil \frac{m-s}{2} \rceil)$. Solving this inequality and noticing that $\sigma(s)$ is the least m for which it holds, one obtains that $\sigma(s) \geq 2s - 3$ if $s + \sigma(s)$ is even and $\sigma(s) \geq 2s - 1$ if $s + \sigma(s)$ is odd.
The inequality used above is a simple application of Hall's theorem, in line with the pairing strategies for Breaker outlined in [2]. Namely, if $m \leq \frac{4}{3}(s + \lceil \frac{m-s}{2} \rceil)$, Breaker may pair each vertex to $2(m - s) + 2$ edges, and splitting these into $m - s + 1$ different pairs at each vertex he plays one edge in each pair when Maker plays the other. Then Maker cannot play more than $m - (m - s + 1) = s - 1$ edges at any vertex so Breaker wins the game with parameters (m, s). $\square$

However, better lower bounds are expected to exist, which will be of more use in practice, since large Shutout Numbers suggest the existence of strategies for Breaker on graphs of smaller maximal degree. Hence we formulate the following conjectures:

Conjecture 3.2 *There are constants* $c, r > 0$ *such that* $\sigma(s) \geq cr^s$.

More weakly, we certainly feel the following should be true:

Conjecture 3.3 *There are constants* $c > 0$ *and* $r > 1$ *such that* $\sigma(s) \geq cs^r$.

It would be of interest to compute small Shutout Numbers. We expect that for the game on parameters $(7, 6)$ the strategy outlined above will prove useless. In particular:

Conjecture 3.4 *The Shutout Number satisfies* $\sigma(4) \geq 8$.

4. References

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Department of Mathematics, University College London, UK
E-mail address: a.cioba.12@ucl.ac.uk

Department of Mathematics, Stanford University, USA
E-mail address: msavvas@math.stanford.edu